(2)

MARYLAND

COLLEGE PARK CAMPUS

Locking Effects in the Finite Element Approximation of Elasticity Problems

by

Ivo Babuška

and

Manil Suri

Technical Note BN-1119

December 1990

INSTITUTE FOR PHYSICAL SCIENCE AND TECHNOLOGY

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Note BN-1119	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Locking Effects in the Finite Element Approximation of Elasticity Problems		5. TYPE OF REPORT & PERIOD COVERED Final life of contract
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Ivo Babuska[2] - Manil Suri[2]		8. CONTRACT OR GRANT NUMBER(s) [1]ONR N00014-89-J-1030 [2]AFOSR 89-0252
9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute for Physical Science and Technology University of Maryland College Park, MD 20742-2431		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Office of Naval Research Arlington, VA 22217		12. REPORT DATE December 1990
		13. NUMBER OF PAGES 40
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release: distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper, we analyze the phenomenon of "Poisson Locking". This is said to occur when the approximations obtained using the finite element method for elasticity problems deteriorate as a result of the Poisson ratio being close to O.5. Using the general theory developed in an earlier paper on locking, we give precise mathematical definitions of locking and robustness and apply some abstract theorems to analyze the locking of various h, p and h-p finite element schemes. In particular, we analyze two types of rectangular elements for the h-version. Extensions to the three-dimensional case are also discussed.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Locking effects in the finite element approximation of elasticity problems.

by

Ivo Babuška[*]
Institute of Physical Science and Technology
University of Maryland
College Park, Maryland 20742

and

Manil Suri[†]
Department of Mathematics and Statistics
University of Maryland, Baltimore County
Baltimore, Maryland 21228

December 1990

MD90-56-IBMS

TR90-56

[*]The work of this author was supported in part by the Office of Naval Research under Naval Research Grant N00014-90-J-1030.

[†]The work of this author was supported in part by the Air Force Office of Scientific Research, Air Force Systems Command, U.S. Air Force, under grant AFOSR 89-0252.

1. Introduction.

"Locking" is a phenomenon associated with the numerical approximation of certain problems whose mathematical formulations involve a parameter dependency. The problem we consider here is the analysis of elastic materials with the parameter being ν, the Poisson ratio. For ν close to 0.5 (i.e., when the material is nearly incompressible), it is well known that various finite element schemes (for example, the h-version using piecewise linear polynomials on a triangular mesh) result in poor observed convergence rates in the displacements, for practical ranges of the discretization. This is due to locking (called *dilatation* or *Poisson* locking in engineering). It occurs because for the limiting case $\nu = 0.5$, the exact solution $\vec{u}$ must satisfy the constraint

$$\operatorname{div} \vec{u} = 0. \tag{1.1}$$

The imposition of (1.1) on the approximation as well is what leads to locking in this example.

There are several other problems where similar locking effects may be observed -- for example, in plate and shell models, where "shear" and "membrane" locking occur when the thickness "t" is very small and in heat transfer through anisotropic materials where locking occurs when the ratio of conductivities in different directions is close to zero. For problems involving locking, see [4] and Section 53 of [15].

Various methods have been suggested to overcome the effects of locking. One possibility is the use of *mixed methods*, which involve reformulating the problem in a special way. Examples of mixed methods that have been suggested to overcome Poisson ratio locking may be found in [11], among others. An advantage of these methods is that they generally yield good approximations to the "pressures" as well.

Here, we shall concentrate solely on Poisson locking when the accuracy of the displacements and energy (and not the pressures) is of interest. Our goal is to investigate the *robustness* of several finite element approximations using the standard (as opposed to mixed) formulation (also called the displacement formulation). By a *robust* scheme, we mean one which leads to acceptable error levels using a practical range of discretization, no matter how close the parameter is to its limiting value. The use of the standard formulation avoids the special reformulations required by mixed methods and in practice could be the only one available in the context of various commercial codes. Hence it is particularly useful to investigate the associated locking and robustness properties. In this connection, the accurate recovery of the pressures may be accomplished through various post-processing techniques (see [22], for example).

In [4], we have developed a general mathematical theory for locking and robustness and their quantitative assessment. We use this theory here to analyze Poisson locking. Accordingly, in Section 2, we prove some required regularity results and in Section 3, we adapt various definitions and theorems from [4] to the problem at hand. In this paper, we restrict ourselves to the case of triangular and parallelogram quasiuniform meshes. The case of curved elements, which is particularly important in the context of the p-version, is discussed in [5].

Section 5 contains various locking and robustness results for the h-version. In [19] (see also [20]), it was shown that no locking results when polynomials of degree $p \geq 4$ are used on triangular meshes. We present an alternate proof here and also give some results for $p < 4$. The results of [19] were restricted to triangular elements. Here we investigate the use of two types of rectangular elements as well and show that locking cannot be

avoided in either case, for any p. Next, in Section 6, we indicate how one can show optimal rates of convergence in the displacements for the p and h-p version uniformly in ν. Our approach (and definition of locking) is different from that of [23] and [19], by which one only gets optimality up to an arbitrary $\varepsilon > 0$. Section 7 contains extensions of our theory to general 3-d analogs of Poisson locking.

2. Regularity Results.

Let $\Omega \subset \mathbb{R}^2$, $0 \in \Omega$ be a bounded, simply connected, polygonal domain with boundary $\Gamma = \sum_{i=1}^{M} \bar{\Gamma}_i$, where Γ_i are open straight line segments with internal angles > 0. For $S \subset \mathbb{R}^n$, we will denote by $H^r(S)$ the usual Sobolev spaces (r real) with $\|\cdot\|_{r,S}$ and $|\cdot|_{r,S}$ denoting the corresponding norm and seminorm respectively. For any space V, $\vec{V}$ will denote $V \times V$ (the norm of V and $\vec{V}$ will be denoted by the same symbol). We will use $C^{(k)}(S)$, $k \geq 0$ integer to denote the usual set of functions on $\bar{S}$ with k continuous derivatives, and $\|\cdot\|_{C^{(k)}(S)}$ to denote its norm (the superscript k being omitted when 0).

The problem we are interested in is the elasticity problem given by the following Lamé-Navier equations

$$(2.1)\qquad -\Delta^*_\nu \vec{u}_\nu = \frac{-E}{2(1+\nu)}\Delta\vec{u}_\nu - \frac{E}{2(1+\nu)(1-2\nu)}\,\mathrm{grad}\,\mathrm{div}\,\vec{u}_\nu = \vec{f} \quad \text{in } \Omega$$

$$(2.2)\qquad T_\nu(\vec{u}) = \vec{g} \quad \text{on } \Gamma$$

where $\vec{u}_\nu = (u_1, u_2)$ and where the tractions $(T_\nu(\vec{u}_\nu))_i$ are given for $i = 1,2$ by

$$(2.3)\qquad (T_\nu(\vec{u}_\nu))_i = \left[\frac{E}{1+\nu}\right] \sum_{j=1}^{2} \left[\varepsilon_{ij}(\vec{u}_\nu) + \delta_{ij}\frac{\nu}{1-2\nu}\,\mathrm{div}\,\vec{u}_\nu\right] n_j.$$

Here, (n_1, n_2) is the unit outward normal to Γ and $\{\varepsilon_{ij}\}$ is the strain tensor given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right).$$

The coefficient $0 \le \nu < 0.5$ represents the Poisson ratio and E the modulus of elasticity, which are related to the Lamé constants λ, μ by

$$(2.4) \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}.$$

We assume that

$$(2.5) \qquad \iint_\Omega \vec{f} \cdot \vec{R}\, dx + \int_\Gamma \vec{g} \cdot \vec{R}\, ds = 0$$

for any rigid body motion $\vec{R}$ to ensure that (2.1) and (2.2) have a solution (unique up to $\vec{R}$).

As usual, the components σ^ν_{ij}, $i = 1,2$ of the stress tensor are then given by

$$\sigma^\nu_{11} = \lambda\theta^\nu + 2\mu \frac{\partial u_1}{\partial x_1}$$

$$\sigma^\nu_{22} = \lambda\theta^\nu + 2\mu \frac{\partial u_2}{\partial x_2}$$

$$\sigma^\nu_{12} = \mu\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)$$

where $\theta^\nu = \operatorname{div} \vec{u}_\nu$.

We have used the index ν to emphasize the dependence on ν. If no ambiguity can occur, we will omit this index.

We will assume, for simplicity (but without loss of generality), that $E = 1$. We will also consider the following variational form instead of (2.1) - (2.2): Find, for given $\nu \in [0, 0.5)$, a $\vec{u}_\nu \in \vec{H}^1(\Omega)$ satisfying, $\forall \vec{v} \in \vec{H}^1(\Omega)$,

$$(2.6) \qquad B_\nu(\vec{u}_\nu,\vec{v}) = a_\nu(\vec{u}_\nu,\vec{v}) + \frac{\nu}{(1+\nu)(1-2\nu)}(\mathrm{div}\,\vec{u}_\nu,\ \mathrm{div}\,\vec{v}) = F(\vec{v})$$

where

$$(2.7) \qquad a_\nu(u,v) = \frac{1}{1+\nu}\iint_\Omega \sum_{i,j=1}^{2} \varepsilon^\nu_{ij}(\vec{u})\,\varepsilon^\nu_{ij}(\vec{v})\,dx$$

$$(2.8) \qquad F(\vec{v}) = \iint_\Omega \vec{f}\cdot\vec{v}\,dx + \int_\Gamma \vec{g}\cdot\vec{v}\,ds.$$

We denote the problem (2.6) by P_ν. It can be assumed equivalent to (2.1) - (2.2) without loss of generality.

Corresponding to (2.6), we define the energy norm by

$$\|\vec{u}\|^2_{E,\nu} = B_\nu(\vec{u},\vec{u}), \qquad \vec{u}\in \vec{H}^1(\Omega)\ .$$

Using Korn's inequality, we see that

$$(2.9) \qquad C_1\|\vec{u}\|_{1,\Omega} \le \|\vec{u}\|_{E,\nu} \le C_2(1-2\nu)^{-1/2}\,\|\vec{u}\|_{1,\Omega}$$

where $\|\cdot\|_{1,\Omega}$ denotes the norm in the quotient space $\vec{H}^1(\Omega)\backslash\{$Rigid Body motions$\}$ and C_1 and C_2 are constants independent of ν. Obviously, for ν bounded away from 0.5, the two norms are equivalent.

For any $k \ge 1$, $0 \le \nu < 0.5$, let us define the space $H_{k,\nu} = H_{k,\nu}(\Omega)$ furnished with the weighted norm (modulo rigid body motions),

$$(2.10) \qquad \|\vec{u}\|^2_{k,\nu} = \|\vec{u}\|^2_{k,\Omega} + (1-2\nu)^{-2}\|\mathrm{div}\,\vec{u}\|^2_{k-1,\Omega}.$$

Related to this definition are the characterizations of balls $H^B_{k,\nu}$, $(B\in R^+)$ given by

$$H^B_{k,\nu} = \{\vec{u}\in \vec{H}^k(\Omega),\ \ \|\vec{u}\|_{k,\nu}\le B\}$$

The spaces $H_{k,\nu}$ are the natural spaces to consider while

characterizing the solutions of (2.6) (or (2.1) - (2.2)). Their choice is motivated by the following theorem from [23].

Theorem 2.1. Let Ω be a smooth domain with smooth boundary Γ. Let k be an integer ≥ 1. If $\vec{u}_\nu \in \vec{H}^1(\Omega)$ denotes the solution to (2.1) - (2.2) for data $\vec{f} \in H^{k-2}(\Omega)$, $\vec{g} \in H^{k-3/2}(\Gamma)$, then $\vec{u}_\nu \in H^B_{k,\nu}$ with

$$B = C\left\{\|\vec{f}\|_{k-2,\Omega} + \|\vec{g}\|_{k-3/2,\Gamma}\right\}$$

where C depends on k but is independent of ν.

For the case that the domain is a polygon, the above theorem will again be valid, but only for a restricted range of k, i.e., $k \leq k_0$, where k_0 is determined by the domain. We now prove

Lemma 2.1. Let $\vec{u}_\nu \in H^B_{k,\nu}$ for $k \geq 2$, $0 \leq \nu < 0.5$. Let $\vec{f}$ and $\vec{g}$ be defined in terms of $\vec{u}_\nu$ by (2.1), (2.2) respectively. Then $\vec{f}$ and $\vec{g}$ satisfy (2.5) and

$$\|\vec{f}\|_{k-2,\Omega} + \sum_i \|\vec{g}\|_{k-3/2,\Gamma_i} \leq CB \tag{2.11}$$

where C is a constant independent of ν, $\vec{u}_\nu$ and B.

Proof. Letting $\vec{v}$ in (2.6) be a rigid body motion, it is easy to see that (2.5) is satisfied. From (2.1), we see that $(E = 1)$

$$\begin{aligned}\|\vec{f}\|_{k-2,\Omega} &\leq \frac{1}{2(1+\nu)}\|\Delta\vec{u}_\nu\|_{k-2,\Omega} + \frac{1}{2(1+\nu)(1-2\nu)}\|\operatorname{grad}\operatorname{div}\vec{u}_\nu\|_{k-2,\Omega} \\ &\leq \frac{1}{2}\left[\|\vec{u}_\nu\|_{k,\Omega} + \frac{1}{1-2\nu}\|\operatorname{div}\vec{u}_\nu\|_{k-1,\Omega}\right] \\ &\leq \frac{\sqrt{2}}{2}\|\vec{u}_\nu\|_{k,\nu} \leq \frac{\sqrt{2}}{2}B.\end{aligned}$$

Similarly, by (2.2), (2.3),

$$\sum_{i} \|\vec{g}\|_{k-3/2,\Gamma_i} \le \frac{C}{1+\nu} \sum_{i} \left(\|\vec{u}_\nu\|_{k-1/2,\Gamma_i} + \frac{\nu}{1-2\nu}\|\operatorname{div}\vec{u}_\nu\|_{k-3/2,\Gamma_i}\right)$$

$$\le C\|\vec{u}_\nu\|_{k,\nu} \le CB$$

which gives (2.11). □

We will be interested in the limiting sets

$$H^B_{k,L} = \{\vec{u} \in H^B_{k,\nu} \qquad \forall \nu \in [0,\ 0.5)\}\ .$$

Obviously, these may be equivalently characterized as $H^B_{k,L} = H^B_k \cap H_{k,L}$ where

$$H^B_k = \{\vec{u} \in \vec{H}^k(\Omega),\quad \|\vec{u}\|_{k,\Omega} \le B\}$$

$$H_{k,L} = \{\vec{u} \in \vec{H}^k(\Omega),\quad \operatorname{div}\vec{u} = 0\}\ .$$

Let us prove the following Lemma.

<u>Lemma 2.2</u>. Let $\vec{u}_\nu \in H^B_{k,\nu}$, $k \ge 2$, $0 \le \nu < 0.5$ be such that is satisfies (2.1) with $\vec{f} = \vec{0}$. Then there exists a $\vec{u}_L \in H^{(\alpha B)}_{k,L}$ such that

(2.12) $$\|\vec{u}_\nu - \vec{u}_L\|_{k,\Omega} \le \gamma B(1-2\nu)$$

where the constants α and γ are independent of ν, $\vec{u}_\nu$ and B.

<u>Proof</u>. Since $\vec{u}_\nu$ satisfies (2.1) with $\vec{f} = 0$, the stresses σ_{ij} relate to the Airy biharmonic function U in the usual fashion,

(2.13) $$\sigma_{11} = \frac{\partial^2 U}{\partial x_2^2},\qquad \sigma_{12} = -\frac{\partial^2 U}{\partial x_1 \partial x_2},\qquad \sigma_{22} = \frac{\partial^2 U}{\partial x_1^2}.$$

Let us denote (up to an arbitrary constant)

(2.14) $$P = \Delta U\ .$$

Then since U is biharmonic, P is harmonic. Denoting by Q the harmonic

conjugate of P and using the fact that Ω is simply connected, we have for any $s \geq 1$,

$$|P|_{s,\Omega} = |Q|_{s,\Omega}. \tag{2.15}$$

Also, with $z = x_1 + ix_2$,

$$h(z) = P(x_1,x_2) + iQ(x_1,x_2) \tag{2.16}$$

is a holomorphic function on Ω. We define

$$\phi(z) = \frac{1}{4}\int_0^z h(z)\,dz = p_1(x_1,x_2) + ip_2(x_1,x_2). \tag{2.17}$$

Then (see eq. (30.8) of [17]), the solution to (2.1) - (2.2) is given (up to rigid body motion) by

$$u_\nu^i = C_1(\nu)\frac{\partial U}{\partial x_i} + C_2(\nu)p_i \tag{2.18}$$

where

$$C_1(\nu) = -(1+\nu), \qquad C_2(\nu) = 4(1+\nu)(1-\nu). \tag{2.19}$$

By equation (30.7) of [17], we have

$$\frac{\partial p_i}{\partial x_i} = \frac{P}{4} \qquad i = 1,2$$

so that using (2.18) and (2.14),

$$\operatorname{div}\vec{u}_\nu = \left[C_1(\nu) + \frac{C_2(\nu)}{2}\right]P$$

$$= (1-2\nu)(1+\nu)P.$$

Hence,

$$\|P\|_{k-1,\Omega} \leq \frac{1}{(1-2\nu)(1+\nu)}\|\operatorname{div}\vec{u}_\nu\|_{k-1,\Omega} \leq \|\vec{u}_\nu\|_{k,\nu} \leq B.$$

Therefore, using (2.15) - (2.16), for $k \geq 2$,

$$\|h\|_{k-1,\Omega} \leq CB$$

(where the norm is a quotient norm modulo constants).

Then, by (2.17), for $i = 1,2$ (modulo rigid body motions),

$$\|p_i\|_{k,\Omega} \le CB, \tag{2.20}$$

so that by (2.18) - (2.20),

$$\left\|\frac{\partial U}{\partial x_i}\right\|_{k,\Omega} \le \frac{1}{1+\nu}\|u_\nu^i\|_{k,\Omega} + 4(1-\nu)\|p_i\|_{k,\Omega} \tag{2.21}$$

$$\le \|\vec{u}_\nu\|_{k,\nu} + 4\|p_i\|_{k,\Omega}$$

$$\le CB\,.$$

We now define $\vec{u}_L$ (up to rigid body motion) by

$$u_L^i = C_1(0.5)\frac{\partial U}{\partial x_i} + C_2(0.5)\,p_i, \qquad i = 1,2\,. \tag{2.22}$$

Then $\operatorname{div}\vec{u}_L = 0$ and by (2.20) - (2.22),

$$\|\vec{u}_L\|_{k,\Omega} \le \alpha B$$

so that $\vec{u}_L \in H_{k,L}^{(\alpha B)}$. Also, by (2.18), (2.22),

$$u_\nu^i - u_L^i = (1-2\nu)\left[\frac{1}{2}\frac{\partial U}{\partial x_i} + (1+2\nu)p_i\right] \tag{2.23}$$

so that by (2.20), (2.21), (2.23),

$$\|\vec{u}_\nu - u_L\|_{k,\Omega} \le \gamma B(1-2\nu),$$

which establishes the lemma. □

We now extend the above lemma to the case that $\vec{f} \ne \vec{0}$, to get the following theorem.

<u>Theorem 2.2</u>. Given $\vec{u}_\nu \in H_{k,\nu}^B$, $k \ge 2$, $0 \le \nu < 0.5$, there exists a $\vec{u}_L \in H_{k,L}^{(\alpha B)}$ such that (2.12) holds with α,γ being constants independent of ν, $\vec{u}_\nu$ and B.

Proof. Let $\vec{u}_\nu \in H^B_{k,\nu}$, $k \geq 2$ be given. Define $\vec{f}$ by (2.1). We will reduce this case to the case of Lemma 2.2, where $\vec{f} = \vec{0}$.

We first find a particular solution of (2.1). Since by Lemma 2.1, $\vec{f} \in \vec{H}^{k-2}(\Omega)$ for some $k \geq 2$, we may find an extension $\vec{F}$ of $\vec{f}$ to the whole of $\vec{R}$ such that $\vec{F}$ has compact support, satisfies

(2.24) $$\|\vec{F}\|_{k-2,\mathbb{R}^2} \leq C\|\vec{f}\|_{k-2,\Omega} \leq C\tau B$$

(by Lemma 2.1) and also satisfies the compatibility condition

(2.25) $$\iint_{\mathbb{R}^2} \vec{F} \cdot \vec{R} = 0$$

for any rigid body motion $\vec{R}$. Then the problem (2.1) with $\vec{f}$ replaced by $\vec{F}$ will have a unique solution over $\mathbb{R}^2$ (up to rigid body motion), which we denote by $\vec{W}_\nu$. Obviously, $\vec{W}_\nu|_\Omega$ is a particular solution for (2.1).

Now, let $\hat{W}_{\nu,i}$, $i = 1,2$ denote the Fourier transform of $(\vec{W}_\nu)_i$ and $\hat{F}_i$ the Fourier transform of F_i. Then if ξ_1, ξ_2 represent the transformed variables, we have by (2.1),

$$\frac{1}{2(1+\nu)}\left[\left(\xi_1^2+\xi_2^2\right)\hat{W}_{\nu,1} + \frac{1}{1-2\nu}\left(\xi_1^2\hat{W}_{\nu,1} + \xi_1\xi_2\hat{W}_{\nu,2}\right)\right] = \hat{F}_1$$

$$\frac{1}{2(1+\nu)}\left[\left(\xi_1^2+\xi_2^2\right)\hat{W}_{\nu,2} + \frac{1}{1-2\nu}\left(\xi_2^2\hat{W}_{\nu,2} + \xi_1\xi_2\hat{W}_{\nu,1}\right)\right] = \hat{F}_2$$

which gives

(2.26) $$\begin{bmatrix}\hat{W}_{\nu,1}\\ \hat{W}_{\nu,2}\end{bmatrix} = \frac{(1+\nu)}{(1-\nu)(\xi_1^2+\xi_2^2)^2}\begin{bmatrix}(1-2\nu)(\xi_1^2+\xi_2^2)+\xi_2^2 & -\xi_1\xi_2\\ -\xi_1\xi_2 & (1-2\nu)(\xi_1^2+\xi_2^2)+\xi_1^2\end{bmatrix}\begin{bmatrix}\hat{F}_1\\ \hat{F}_2\end{bmatrix}$$

Let us put $\nu = 0.5$ in (2.26) to define

(2.27) $$\begin{bmatrix}\hat{W}_{L,1}\\ \hat{W}_{L,2}\end{bmatrix} = \frac{3}{(\xi_1^2+\xi_2^2)^2}\begin{bmatrix}\xi_2^2 & -\xi_1\xi_2\\ -\xi_1\xi_2 & \xi_1^2\end{bmatrix}\begin{bmatrix}\hat{F}_1\\ \hat{F}_2\end{bmatrix}$$

Using (2.26), we see that

$$\widehat{\operatorname{div} \vec{W}_\nu} = -i(\xi_1 \hat{W}_{\nu,1} + \xi_2 \hat{W}_{\nu,2}) \tag{2.28}$$

$$= \frac{-i(1-2\nu)(1+\nu)}{(1-\nu)(\xi_1^2+\xi_2^2)} (\xi_1 \hat{F}_1 + \xi_2 \hat{F}_2)$$

and putting $\nu = 0.5$ in (2.28), we obtain

$$\operatorname{div} \vec{W}_L = 0. \tag{2.29}$$

Now by Parseval's equality,

$$|\vec{W}_\nu|_{1,\Omega} \le |\vec{W}_\nu|_{1,\mathbb{R}^2} = |\hat{W}_\nu|_{1,\mathbb{R}^2} \cdot \tag{2.30}$$

Using (2.26), we have

$$|\hat{W}_\nu|^2_{k,\mathbb{R}^2} \le \sum_{i=1}^{2} |(\xi_1^2+\xi_2^2+ \xi_1\xi_2)\,\hat{W}_{\nu,i}|^2_{k-2,\mathbb{R}^2} \tag{2.31}$$

$$\le C \sum_{i=1}^{2} |\hat{F}_i|^2_{k-2,\mathbb{R}^2} = C|\vec{F}|^2_{k-2,\mathbb{R}^2}$$

$$\le C(\tau B)^2$$

by (2.24). Hence, by (2.30) - (2.31),

$$\|\vec{W}_\nu\|_{k,\Omega} \le C\tau B . \tag{2.32}$$

Similarly,

$$\|\vec{W}_L\|_{k,\Omega} \le C\tau B . \tag{2.33}$$

Also, by considering (2.28), we obtain

$$\|\operatorname{div} \vec{W}_\nu\|_{k-1,\Omega} \le C\tau\,(1-2\nu)B \tag{2.34}$$

so that combining (2.32) and (2.34) gives

$$\vec{W}_\nu \in H^{(\alpha B)}_{k,\nu} \tag{2.35}$$

for some α. Moreover, by (2.29) and (2.33),

$$\vec{W}_L \in H^{(\alpha B)}_{k,L} . \tag{2.36}$$

Next, using (2.26) and (2.27), we have

$$\begin{bmatrix} \hat{W}_{\nu,1} - \hat{W}_{L,1} \\ \hat{W}_{\nu,2} - \hat{W}_{L,2} \end{bmatrix} = \frac{1-2\nu}{(1-\nu)(\xi_1^2+\xi_2^2)^2} \begin{bmatrix} (1+\nu)(\xi_1^2+\xi_2^2)-2\xi_2^2 & 2\xi_1\xi_2 \\ 2\xi_1\xi_2 & (1+\nu)(\xi_1^2+\xi_2^2)-2\xi_1^2 \end{bmatrix} \begin{bmatrix} \hat{F}_1 \\ \hat{F}_2 \end{bmatrix}$$

so that using an argument similar to (2.31), we have for some τ,

$$\|\vec{W}_\nu - \vec{W}_L\|_{k,\Omega} \le \tau B(1-2\nu) . \tag{2.37}$$

Now let on Ω

$$\vec{w}_\nu = \vec{u}_\nu - \vec{W}_\nu . \tag{2.38}$$

Then by (2.35), $\vec{w}_\nu \in \vec{H}^{(\alpha B)}_{k,\nu}$ for some α. Moreover, $\vec{w}_\nu$ satisfies (2.1) with $\vec{f} = \vec{0}$. Hence, applying Lemma 2.2, there exists a $\vec{w}_L \in H^{(\alpha B)}_{k,L}$ such that

$$\|\vec{w}_\nu - \vec{w}_L\|_{k,\Omega} \le \gamma B(1-2\nu). \tag{2.39}$$

Finally, taking

$$\vec{u}_L = \vec{W}_L + \vec{w}_L \tag{2.40}$$

we see (using (2.36)) $u_L \in H^{(\alpha B)}_{k,L}$. Also by (2.38), (2.40)

$$\begin{aligned} \|\vec{u}_\nu - \vec{u}_L\|_{k,\Omega} &= \|(\vec{W}_\nu + \vec{w}_\nu) - (\vec{W}_L + \vec{w}_L)\|_{k,\Omega} \\ &\le \|\vec{W}_\nu - \vec{W}_L\|_{k,\Omega} + \|\vec{w}_\nu - \vec{w}_L\|_{k,\Omega} \\ &\le \gamma B(1-2\nu) \end{aligned}$$

for some γ, using (2.37), (2.39). This proves the theorem. $\square$

3. Locking and robustness.

We now discuss the approximate solution of (2.6). Let $\{\vec{V}^N\}$ be a sequence of finite dimensional subspaces of $\vec{H}^1(\Omega)$ (N denoting the dimension, $N \in \mathcal{N}$). We then find $\vec{u}^N_\nu \in \vec{V}^N$ satisfying

$$B_\nu(\vec{u}^N_\nu, \vec{v}) = B_\nu(\vec{u}_\nu, \vec{v}) \qquad \forall \vec{v} \in \vec{V}^N. \tag{3.1}$$

(3.1) immediately gives

$$\|\vec{u}_\nu - \vec{u}^N_\nu\|_{E,\nu} \le \inf_{\vec{w}\in\vec{V}^N} \|\vec{u}_\nu - \vec{w}\|_{E,\nu}. \tag{3.2}$$

The sequence $\{\vec{V}^N_\nu\}$ defines an *extension procedure* $\mathcal{F}$, i.e., a rule to increase the dimension N (and thereby decrease the error in (3.2)).

We will restrict our attention to the case when the exact solutions $\vec{u}_\nu$ belong to the sets $H_{k,\nu} \subset \vec{H}^k(\Omega)$ $(k \ge 2)$ introduced in the previous section. We assume that the sequence $\mathcal{F} = \{\vec{V}^N\}$ is such that for any $0 \le \nu < 0.5$,

$$C_1F_0(N) \le \sup_{\vec{w}\in\vec{H}^B_k} \inf_{\vec{v}\in\vec{V}^N} \|\vec{w} - \vec{v}\|_{1,\Omega} \le C_2F_0(N) \tag{3.3}$$

where $F_0(N)\longrightarrow 0$ as $N\longrightarrow\infty$, F_0 independent of ν, and C_1, C_2 independent of N and ν.

Let $0 < \nu_0 < 0.5$ be bounded away from 0.5. Then using (2.9), (3.2) and (3.3), we see that the following will hold uniformly for all $0 \le \nu \le \nu_0$

$$C_1(\nu_0)F_0(N) \le \sup_{\vec{u}_\nu\in H^B_{k,\nu}} E_\nu(\vec{u}_\nu - \vec{u}^N_\nu) \le C_2(\nu_0)F_0(N) \tag{3.4}$$

where

$$E_\nu(\vec{w}) = \|\vec{w}\|_{1,\Omega} \quad \text{or} \quad \|\vec{w}\|_{E,\nu} \tag{3.5}$$

and C_1, C_2 now depend on ν_0.

A procedure $\mathcal{F}$ for which the estimate (3.4) holds uniformly for all $0 \le \nu < 0.5$ will be called *free from locking* for the sets $H_{k,\nu}$ with respect to the E_ν measure. We make this more precise by using definition 3.1 below, which has been adapted from [4], in which a more general treatment may be found (for e.g., we could formulate the question of locking in terms of other error measures and solution sets different from the ones considered here).

For $\nu \in [0, 0.5)$ and $N \in \mathcal{N}$, we define the *locking ratio* with respect to the spaces $H_{k,\nu} \subset \vec{H}^k(\Omega)$ and error measures $\{E_\nu\}$ (as in (3.5)) for the problems (3.1) by

$$L(\nu, N) = \sup_{\vec{u}_\nu \in H^B_{k,\nu}} E_\nu(\vec{u}_\nu - \vec{u}^N_\nu)(F_0(N))^{-1}. \tag{3.6}$$

We then have

<u>Definition 3.1</u>. The extension procedure $\mathcal{F}$ is *free from locking* for the family of problems (3.1), $\nu \in [0, 0.5)$ with respect to the solution sets $H_{k,\nu} \subset \vec{H}^k(\Omega)$ and error measures E_ν if and only if

$$\operatorname*{Lim\,sup}_{N \to \infty} \left\{ \sup_{\nu \in [0,\, 0.5)} L(\nu, N) \right\} = M < \infty .$$

$\mathcal{F}$ shows locking of order $f(N)$ if and only if

$$0 < \operatorname*{Lim\,sup}_{N \to \infty} \left\{ \sup_{\nu} L(\nu, N) \frac{1}{f(N)} \right\} = C < \infty$$

where $f(N) \to \infty$ as $N \to \infty$. It shows locking of *at least* (respectively *at most*) order $f(N)$ if $C > 0$ (respectively $C < \infty$).

Related to the concept of locking is the concept of *robustness*, which we define as follows:

Definition 3.2. The extension procedure $\mathcal{F}$ is *robust* for the family of problems (3.1), $\nu \in [0, 0.5)$ with respect to the solution sets $\vec{H}_{k,\nu} \subset \vec{H}^k(\Omega)$ and error measures E_ν if and only if

$$\lim_{N\to\infty} \sup_{\nu} \sup_{\vec{u}_\nu \in H^B_{k,\nu}} E_\nu(\vec{u}_\nu - \vec{u}^N_\nu) = 0 .$$

It is robust with uniform order $g(N)$ if and only if

$$\sup_{\nu} \sup_{\vec{u}_\nu \in H^B_{k,\nu}} E_\nu(\vec{u}_\nu - \vec{u}^N_\nu) \le g(N)$$

where $g(N) \to 0$ as $N \to \infty$.

The relationship between the above two definitions is given by the following theorem, from [4].

Theorem 3.1. $\mathcal{F}$ is free from locking if and only if it is robust with uniform order $F_0(N)$. Moreover, let $f(N)$ be such that

$$f(N)F_0(N) = g(N) \to 0 \quad \text{as} \quad N \to \infty.$$

Then $\mathcal{F}$ shows locking of order $f(N)$ if and only if it is robust with uniform order $g(N)$.

Note that $\mathcal{F}$ is non-robust if and only if it shows locking of order $(F_0(N))^{-1}$.

Let us make some comments about the meaning of the notions we have introduced. The inequality (3.3) characterizes the approximation properties of the space $\vec{V}^N$ with respect to the set $\vec{H}^B_k$, i.e., the smallest error which *could* be achieved (for the most unfavorable $\vec{w}$ in $\vec{H}^B_k$). The locking ratio characterizes the ratio of the accuracy of the finite element solution

(for the most unfavorable exact solution in $\vec{H}_{k,\nu}$) to the best accuracy which could be achieved in the sense of (3.3), using the finite element space V^N. Hence an extension procedure (used to construct a sequence of finite element solutions) is called free of locking if for all $\vec{u}_\nu \in \vec{H}^B_{k,\nu}$, $\nu \in [0, 0.5)$, it leads to a rate of convergence which is asymptotically the same as the best rate which the set V^N could provide (in the sense of (3.3)). The order $f(N)$ of locking characterizes the asymptotic strength of the locking, i.e. the convergence rate of the finite element solution compared to the best possible rate. The robustness gives a measure of the rate of convergence which holds independent of the parameter ν.

Remark 3.1. In [19], [20], [23], the condition used to obtain uniform rates of convergence in ν (i.e., characterize the absence of locking) is

$$(3.7) \qquad \|\vec{u}_\nu - \vec{u}^N_\nu\|_{1,\Omega} \le C \inf_{\vec{v}\in\vec{V}^N} \|\vec{u}_\nu - \vec{v}\|_{1,\Omega} \qquad \forall \nu \in [0, 0.5)$$

where C is a constant independent of N, ν. Here, $\vec{u}_\nu$, $\vec{u}^N_\nu$ are solutions corresponding to fixed $\vec{f}$ and $\vec{g}$, with different ν. (3.7) is a stronger condition than the one in Definition 3.1 -- it is equivalent to the so-called "divergence stability" and is necessary and sufficient for the absence of locking when *both* the displacements and pressures are considered together. It can be characterized using Definition 3.1 by choosing

$$E_\nu(\vec{w}) = \|\vec{w}\|_{1,\Omega} + \frac{1}{1-2\nu} \|\mathrm{div}\,\vec{w}\|_{0,\Omega}$$

in the locking ratio (3.6). Alternately, one could re-define the locking ratio (3.6) as

$$L(\nu,N) = \sup_{\vec{u}_\nu \in H^B_{k,\nu}} \left\{ \frac{E_\nu(\vec{u}_\nu - \vec{u}^N_\nu)}{\inf_{\vec{v}\in\vec{V}^N} E_\nu(\vec{u}_\nu - \vec{v})} \right\}$$

and then use Definition 3.1 (with $E_\nu(\vec{w}) = \|\vec{w}\|_{1,\Omega}$).

As mentioned in the introduction, we restrict our investigation here to the displacements alone, with Definition 3.1 (and not (3.7)) characterizing the absence of locking. This leads to a relaxation of a condition needed in a theorem from [19] (see Theorem 5.4) and an improvement in the robustness result from [23] (see Section 6). $\square$

As analyzed in [4], the question of locking may be reduced to one of approximability alone if the problem and solution sets satisfy a certain condition, called condition (α). This condition requires that for any $\vec{u}_\nu \in \vec{H}^B_{k,\nu}$, there exist a $\vec{u}_0 \in H^{B'}_{k,L}$ (for some B' independent of $\vec{u}_\nu$, ν; $\vec{u}_0$ depending on $\vec{u}_\nu$) such that

$$\|\vec{u}_\nu - \vec{u}_0\|_{k,\Omega} \le C(1-2\nu)^{1/2}$$

with C a constant independent of ν and $\vec{u}_\nu$. Essentially, condition (α) says that the solutions $\vec{u}_\nu$ are close enough to functions $\vec{u}_0$ in the limit space. As shown in [4], this allows us to answer the question of locking for solutions $\vec{u}_\nu$ just by considering the approximability of functions in the limit space and leads to some useful theorems concerning locking and robustness.

For our particular problem, we see that condition (α) is satisfied (with $B' = \alpha B$) by equation (2.12) (Theorem 2.2). Then Theorem 2.4 from [4] holds and can be stated as follows.

<u>Theorem 3.2</u>. Let us consider the family of problems (3.1), $\nu \in [0, 0.5)$ with the solutions sets $H_{k,\nu} \subset \vec{H}^k(\Omega)$, $k \ge 2$. Then the extension procedure $\mathcal{F}$ is free from locking with respect to the $\vec{H}^1(\Omega)$ norm if and only if it is free with respect to the energy norm. It shows locking of order $f(N)$ in the $\vec{H}^1(\Omega)$ norm if and only if it shows locking of order $f(N)$ in the energy

norm.

Theorem 3.2 shows that the locking behavior using the two error measures under consideration is the same. This is important, because for theoretical purposes, the $\vec{H}^1(\Omega)$ norm is easier to work with, while computationally, results obtained by programs like MSC/PROBE are often in the energy norm (see [5]). We shall therefore only refer to the locking of $\mathcal{F}$, without specifying an error measure.

Let us define, for any space $\vec{V}$,

$$Z(\vec{V}) = \{\vec{u} \in \vec{V}, \quad \operatorname{div} \vec{u} = 0\}.$$

Then we may reduce the question of locking to one involving approximability alone.

<u>Theorem 3.3</u>. For the problems (3.1) with solution sets $H_{k,\nu}$, $k \geq 2$, the extension procedure $\mathcal{F}$ is *robust* with uniform order $g(N)$ given by

$$g(N) = \sup_{\vec{u} \in H^B_{k,L}} \inf_{\vec{w} \in Z(\vec{V}^N)} \|\vec{u} - \vec{w}\|_{1,\Omega}. \tag{3.8}$$

Also, with $F_0(N)$ as in (3.3), $\mathcal{F}$ is free from locking if and only if

$$g(N) \leq CF_0(N). \tag{3.9}$$

It shows locking of order $f(N)$ if and only if

$$C_1F_0(N)f(N) \leq g(N) \leq C_2F_0(N)f(N).$$

<u>Proof</u>. The theorem follows using (2.12) and Theorem 2.2(B) of [4]. □

We see therefore that in order to check for locking, we need only estimate $g(N)$. We now derive an equivalent, more convenient formulation of $g(N)$.

Let for any space $\vec{V}$,

$$W(\vec{V}) = \{\psi \mid \operatorname{Curl} \psi \in \vec{V}\}.$$

Then it is easy to see that

$$Z(\vec{V}) = \{\vec{u} = \text{Curl}\,\psi, \quad \psi \in W(\vec{V})\}\,. \tag{3.9}$$

<u>Lemma 3.1</u>. For $k \geq 1$, $\vec{u} \in H^{B}_{k,L}$ if and only if there exists $\phi \in H^{(\alpha B)}_{k+1}(\Omega) = \{\phi \in H^{k+1}(\Omega),\ \ \|\phi\|_{k+1,\Omega} \leq \alpha B\}$ such that

$$\vec{u} = \text{Curl}\,\phi \tag{3.10}$$

where α is a constant, independent of u, B.

<u>Proof</u>. Let $\phi \in H^{(\alpha B)}_{k+1}(\Omega)$. Then defining $\vec{u}$ by (3.10), we see that $\vec{u} \in H_{k,L}$ with

$$\|\vec{u}\|_{k,\Omega} = \|\text{Curl}\,\phi\|_{k,\Omega} \leq C\|\phi\|_{k+1,\Omega} \leq C\alpha B.$$

Hence $\vec{u} \in H^{B}_{k,L}$.

Conversely, let $\vec{u} \in H^{B}_{k,L}$. Then we may find ϕ satisfying (3.10). Since ϕ is arbitrary up to a constant, we may choose it such that $\int_\Omega \phi = 0$, so that

$$\|\phi\|_{0,\Omega} \leq C|\phi|_{1,\Omega}\,.$$

Then

$$B \geq \|u\|_{k,\Omega} = \|\text{Curl}\,\phi\|_{k,\Omega} \geq C \sum_{s=1}^{k+1} |\phi|_{s,\Omega} \geq C\ \|\phi\|_{k+1,\Omega}$$

so that $\phi \in H^{(\alpha B)}_{k+1}(\Omega)$. □

<u>Theorem 3.4</u>. Let $g(N)$ be as defined in (3.8). Let

$$\tilde{g}(N) = \sup_{\phi \in H^{(\alpha B)}_{k+1}(\Omega)} \ \inf_{\chi \in W(\vec{V}^N)} \|\phi - \chi\|_{2,\Omega}\,. \tag{3.11}$$

Then

$$C_1\tilde{g}(N) \leq g(N) \leq C_2\tilde{g}(N)$$

with C_1, C_2 independent of N.

Proof. Let $\vec{u} \in H^B_{k,L}$ and $\vec{w} \in Z(\vec{V}^N)$. Then by Lemma 3.1 and (3.9) we can find $\phi \in H^{(\alpha B)}_{k+1}(\Omega)$, $\chi \in W(\vec{V}^N)$ such that

$$\vec{u} = \operatorname{Curl}\phi, \quad \vec{w} = \operatorname{Curl}\chi. \tag{3.12}$$

Hence,

$$\|\vec{u} - \vec{w}\|_{1,\Omega} = \|\operatorname{Curl}\phi - \operatorname{Curl}\chi\|_{1,\Omega} \le \|\phi - \chi\|_{2,\Omega}$$

so that

$$g(N) \le C_2\tilde{g}(N).$$

Conversely, let $\phi \in H^{(\alpha B)}_{k+1}(\Omega)$, $\chi \in W(\vec{V}^N)$. Then using Lemma 3.1, let $\vec{u} \in H^B_{k,L}$, $\vec{w} \in Z(\vec{V}^N)$ satisfy (3.12). By adjusting χ by a constant, we can ensure that (3.12) remains true with

$$\|\phi - \chi\|_{0,\Omega} \le C|\phi - \chi|_{1,\Omega}$$

so that

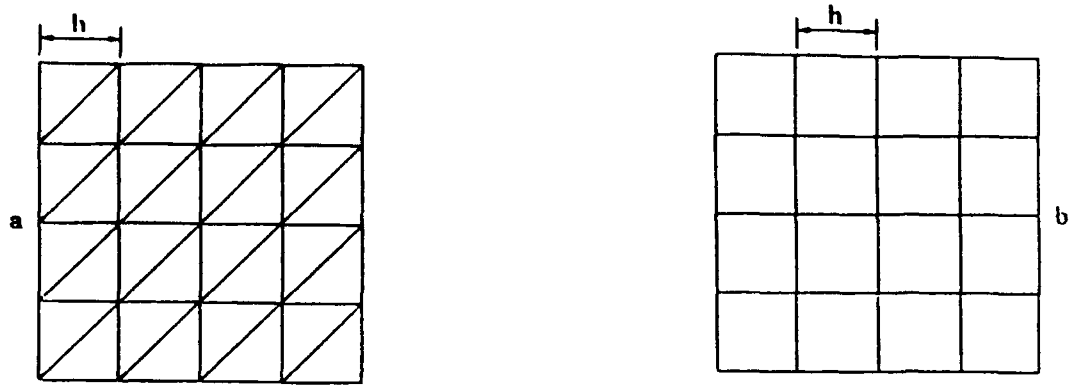
$$\begin{aligned}\inf_{\chi\in W(\vec{V}^N)} \|\phi - \chi\|_{2,\Omega} &\le C \inf_{\chi\in W(\vec{V}^N)} \|\operatorname{Curl}(\phi-\chi)\|_{1,\Omega} \\ &= C \inf_{\vec{w}\in Z(\vec{V}^N)} \|\vec{u} - \vec{w}\|_{1,\Omega}.\end{aligned}$$

The theorem follows. □

4. The extension procedure $\mathcal{F}$.

We now define the subspaces $\{\vec{V}^N\}$. Let $\{\mathcal{T}^h\}$ be a quasiuniform sequence of meshes on Ω (in the usual sense, see [13]), parameterized by h, the mesh spacing. Each $\mathcal{T}^h$ consists of straight-sided parallelogram or triangular elements Ω^h_i, $i = 1,2,\ldots,n(h)$ such that $\bar{\Omega}^h_i \cap \bar{\Omega}^h_j$ is either

empty, a vertex or an entire side for $i \neq j$. We will, in particular, be interested in the case that Ω is a domain which can be covered by *uniform* triangular or rectangular meshes $\mathcal{T}_1^h$, $\mathcal{T}_2^h$ of the type shown (for a unit square) in Figure 4.1.



(a) $\mathcal{T}_1^h$: uniform triangular (b) $\mathcal{T}_2^h$: uniform rectangular

Figure 4.1. Uniform meshes.

The general quasiuniform versions of these meshes will be denoted by $\mathcal{T}_3^h$, $\mathcal{T}_4^h$ respectively. The meaning of quasiuniform triangular $(\mathcal{T}_3^h)$ meshes on a polygonal domain Ω is clear. The quasiuniform meshes $\mathcal{T}_4^h$ will consist of elements Ω_i^h which are rectangles (of possibly different sizes, and not just uniform squares). We will also consider briefly the case of meshes $\mathcal{T}_5^h$ consisting of parallelograms.

For S a triangle or parallelogram, let $P_p(S)$ denote the set of polynomials on S of total degree $\leq p$. For S a parallelogram, we let $Q_p(S)$ be the set of all polynomials on S with degree $\leq p$ in each variable and define

$$Q_p'(S) = P_p(S) \oplus \{x_1^p x_2, x_1 x_2^p\},$$

to be the space of serendipity elements (see [13]).

For any mesh $\mathcal{T}^h$, we now define

$$P^h_{p,-1} = P^h_{p,-1}(\mathcal{T}^h) = \{v \in L_2(\Omega), \quad v|_{\Omega^h_i} \in P_p(\Omega^h_i)\}$$

and for $k \geq 0$,

$$P^h_{p,k} = P^h_{p,k}(\mathcal{T}^h) = P^h_{p,-1} \cap C^{(k)}(\Omega).$$

For rectangular meshes $\mathcal{T}^h$ $(= \mathcal{T}^h_2, \mathcal{T}^h_4,$ or $\mathcal{T}^h_5)$, we analogously define $Q^h_{p,k}(\mathcal{T}^h)$ and $Q^{h\,\prime}_{p,k}(\mathcal{T}^h)$ with $P_p(\Omega^h_i)$ replaced by $Q_p(\Omega^h_i)$, $Q'_p(\Omega^h_i)$ respectively. Then our finite element space $\vec{V}^N = \vec{V}^h_p$ will be taken to be one of $\vec{P}^h_{p,0}$, $\vec{Q}^h_{p,0}$ or $\vec{Q}^{h\,\prime}_{p,0}$.

We now estimate $F_0(N)$ in (3.3).

<u>Theorem 4.1</u>. Let $\mathcal{F} = \{\vec{V}^N\} = \{\vec{V}^h_p\}$ consist of the h-version using a family of quasiuniform meshes $\{\mathcal{T}^h\}$ of parallelograms and triangles, with degree $p \geq 1$ fixed. Then (3.3) is satisfied with $F_0(N)$ given by

$$F_0(N) \sim CN^{-p/2} \qquad \text{for } p < k-1$$
$$\sim CN^{-(k-1)/2} \qquad \text{for } p \geq k-1 \tag{4.1}$$

where C is independent of N but depends on p,k.

<u>Proof</u>. We note that by the usual results (see [13]) for the h-version,

$$F_0(N) \leq CN^{-\min(p,k-1)/2}$$

since $N = O(h^{-2})$. The result for $p \geq k-1$ follows by noting that

$$F_0(N) \geq CN^{-(k-1)/2}$$

using the theory of n-widths [18]. For $p < k-1$, the corresponding lower bound has been proved in Lemma 3.2 of [4].

$\square$

Theorem 4.2. Let $\mathcal{F} = \{\vec{V}^N\} = \{\vec{V}_p^h\}$ consist of the p-version using a fixed mesh $\mathcal{T}^{h_0}$ with increasing polynomial degrees $p \longrightarrow \infty$. Then (3.3) holds, with $F_0(N)$ given by

$$F_0(N) \sim CN^{-(k-1)/2} \tag{4.2}$$

Moreover, as $p \longrightarrow \infty$, (4.2) also holds if the h-p version over a quasi-uniform family of meshes $\{\mathcal{T}^h\}$ is used.

Proof. We refer to [3] for the proof of the p-version and [2] for the proof of the h-p version. □

Remark 4.1. In the case of the h-p version, the following more refined estimate has been established in [2]

$$F_0(N) \sim Ch^{\min(p,k-1)}p^{-(k-1)} \tag{4.3}$$

where $N = N(h,p)$.

Let us now characterize the spaces $W(\vec{V}_p^h)$ for the various choices of $\vec{V}_p^h$. These will be used in the next section.

First, we see that over a single element (T = triangle, S = parallelogram),

$$W\left[\vec{P}_p(T)\right] = P_{p+1}(T) \tag{4.4}$$

$$W\left[\vec{Q}_p(S)\right] = P_{p+1}(S) \cup Q_p(S) \tag{4.5}$$

$$\begin{aligned} W\left[\vec{Q}'_p(S)\right] &= P_{p+1}(S) && \text{for } p \neq 2 \\ &= P_3(S) \cup Q_2(S) = W\left[\vec{Q}_2(S)\right] && \text{for } p = 2 \end{aligned} \tag{4.6}$$

Therefore,

$$W(\vec{P}^h_{p,0}) = P^h_{p+1,1} \tag{4.7}$$

for the triangular meshes $\mathcal{T}^h_1$, $\mathcal{T}^h_3$. Also, for rectangular meshes $\mathcal{T}^h_2$, $\mathcal{T}^h_4$ (and $\mathcal{T}^h_5$),

$$W(\vec{Q}^h_{p,0}) = \left[P^h_{p+1,1} \bigcup Q^h_{p,1}\right] \tag{4.8}$$

and for $p \neq 2$

$$W(\vec{Q}^{h\,\prime}_{p,0}) = P^h_{p+1,1} \tag{4.9}$$

with the case $p = 2$ being given by (4.8).

5. The h-version.

We now analyze the locking and robustness of some h-version extension procedures. Theorem 3.4 will play a key role in our analysis.

We first analyze *triangular* meshes.

Theorem 5.1. Let the extension procedure $\mathcal{F}$ consist of the h-version with piecewise polynomials of degree 1 using a uniform mesh $\mathcal{T}^h_1$ as in Figure 4.1. Then with respect to the solution set $H_{k,\nu}$, $k \geq 2$, $\mathcal{F}$ shows locking of order $N^{1/2}$, i.e., the extension $\mathcal{F}$ is not robust.

Proof. We have $\vec{V}^N = \vec{P}^h_{1,0}$ and by (4.7) $W(\vec{V}^N) = P^h_{2,1}$. By Theorem 3.4, we have to estimate

$$\tilde{g}(N) = \sup_{\Phi\in H^{(\alpha B)}_{k+1}(\Omega)} \inf_{\chi\in P^h_{2,1}(\mathcal{T}^h_1)} \|\Phi - \chi\|_{2,\Omega}.$$

Assume that $\tilde{g}(N) \longrightarrow 0$ as $N \longrightarrow \infty$. Then by the embedding theorem we also have for any $\Phi \in C^\infty(\Omega)$,

$$\inf_{\chi \in P^h_{2,1}(\mathcal{T}^h_1)} \|\Phi - \chi\|_{C(\Omega)} \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty$$

But by Theorem 3 of [7], the set $P^h_{s,1}(\mathcal{T}^h_1)$ is dense in $C(\Omega)$ only if $s \geq 3$. Hence we have a contradiction and the theorem is proven. □

Let us now define

$$\begin{aligned} r(p) &= p \qquad \text{for} \quad p = 2,\ 3,\ 4 \\ &= p+1 \quad \text{for} \quad p \geq 5. \end{aligned}$$

Then we have the following.

<u>Lemma 5.1</u>. Let $\mathcal{T}^h_1$ be the uniform triangular mesh of Figure 4.1. Then for $u \in H^k(\Omega)$, $p \geq 2$,

$$\inf_{v \in P^h_{p,1}(\mathcal{T}^h_1)} \|u - v\|_{0,\Omega} \leq Ch^{\min(k, r(p))} \|u\|_{k,\Omega}. \tag{5.1}$$

Moreover, there exists a function $Q \in C^\infty(\Omega)$ satisfying

$$\inf_{v \in P^h_{p,1}(\mathcal{T}^h_1)} \|Q - v\|_{0,\Omega} \geq Ch^{r(p)}. \tag{5.2}$$

<u>Proof</u>. The result for $p \geq 5$ is standard (see, for e.g., [12]). For $2 \leq p \leq 4$, (5.1) is a generalization of Theorem 4 of [7], and has been proven by C. de Boor [6]. The L_∞ analog of the lower estimate (5.2) was established in [8] for $p = 3$ and [9] for $p = 4$. These results have been recently generalized to the L_q $(1 \leq q \leq \infty)$ case in [10]. □

Lemma 5.1 results in the following lemma. The proof of (5.4) is due to C. de Boor, R. DeVore and A. Ron.

<u>Lemma 5.2</u>. For $\mathcal{T}^h_1$ as above, $u \in H^k(\Omega)$, $p \geq 2$,

$$\inf_{v \in P^h_{p,1}(\mathcal{T}^h_1)} \|u - v\|_{2,\Omega} \leq Ch^{\min(k, r(p))-2} \|u\|_{k,\Omega}. \tag{5.3}$$

Moreover, there exists a function $Q \in C^{\infty}(\Omega)$ satisfying

(5.4) $$\inf_{v \in P^h_{p,1}(\mathcal{T}^h_1)} \|Q - v\|_{2,\Omega} \geq Ch^{r(p)-2}.$$

Proof. (5.3) follows from (5.1) by using an inverse property argument. To prove (5.4), let us choose the same function Q from (5.2) in Lemma 5.1. Suppose $v \in P^h_{p,1}(\mathcal{T}^h_1)$ is such that

$$\|Q - v\|_{2,\Omega} = o(h^{r(p)-2}).$$

Then, by using the above and (5.1), there exists $w \in P^h_{p,1}(\mathcal{T}^h_1)$ such that

$$\begin{aligned} \|(Q - v) - w\|_{0,\Omega} &\leq Ch^2 \|Q - v\|_{2,\Omega} \\ &= o(h^{r(p)}), \end{aligned}$$

which is a contradiction to (5.2). □

Lemma 5.2 gives sharp upper and lower bounds for

$$\tilde{g}(N) = \sup_{\Phi \in H^B_{k+1}(\Omega)} \inf_{\chi \in P^h_{p,1}(\mathcal{T}^h_1)} \|\Phi - \chi\|_{2,\Omega}$$

for $k \geq r(p) - 1$. In fact, we obtain

(5.5) $$C_1 h^{r(p)-2} \leq \tilde{g}(N) \leq C_2 h^{r(p)-2}.$$

Theorems 5.2 - 5.3 below follow immediately from (5.5).

Theorem 5.2. *Let the extension procedure $\mathcal{F}$ consist of the h-version with piecewise polynomials of degree ≤ 2 on $\mathcal{T}^h_1$. Then with respect to the set $H_{k,\nu}$, extension $\mathcal{F}$ is uniformly robust with order $N^{-1/2}$ for $k \geq 2$ and shows locking of $O(N^{1/2})$ for $k \geq 3$.*

Theorem 5.3. Let the extension procedure $\mathcal{F}$ consist of the h-version with piecewise polynomials of degree ≤ 3 on $\mathcal{T}_1^h$. Then with respect to the set $H_{k,\nu}$, extension $\mathcal{F}$ is uniformly robust with order N^{-1} for $k \ge 3$ and shows locking of $O(N^{1/2})$ for $k \ge 4$.

In Theorems 5.1 - 5.3 (for $p \le 3$), we restricted ourselves to the uniform mesh $\mathcal{T}_1^h$. It was essential that the mesh have three pairwise independent directions, since with meshes with additional independent directions, the results will be different. For $p > 3$, we may drop the restriction of uniform meshes and prove the following theorem.

Theorem 5.4. Let the extension procedure $\mathcal{F}$ consist of the h-version with piecewise polynomials of degree $p \ge 4$ on a general quasiuniform mesh $\mathcal{T}_3^h$ Then with respect to the set $H_{k,\nu}$, $k \ge p+1$, $\mathcal{F}$ is uniformly robust with order $N^{-p/2}$ and there is no locking.

Proof. By (4.7), we have $W(\vec{V}^N) = P^h_{p+1,1}$. For $p = 4$, this gives the Argyris triangle, for which the interpolation theory (Theorem 6.1.1 of [13]) shows that

$$\inf_{\chi \in P^h_{p+1,1}} \|\Phi - \chi\|_{2,\Omega} \le Ch^p \|\Phi\|_{p+2,\Omega}. \tag{5.6}$$

Moreover, the proof of Theorem 6.1.1 of [13] may easily be generalized to show that (5.6) holds for any $p \ge 4$. The result follows from this. (Note that the result follows immediately from (5.5) for the mesh $\mathcal{T}_1^h$). □

Remark 5.1. In [19], a more restrictive condition on the mesh is used to prove that locking does not occur for $p \ge 4$. However, the result obtained there is stronger and leads to divergence stability as well, as discussed in Remark 3.1. We avoid the condition on the mesh precisely because our

Definition 3.1 of locking is different, involving only the displacements and not the pressures.

So far we have discussed the triangular mesh $\mathcal{T}_1^h$ (and $\mathcal{T}_3^h$). Let us now discuss the mesh $\mathcal{T}_2^h$ with respect to the extension $\mathcal{F}$ based on $Q_p(S)$. We first prove some auxiliary lemmas.

Lemma 5.3. Let $\Omega_\alpha = \{(x_1,x_2),\ 0 < x_1,x_2 < \alpha\}$, $\frac{1}{2} \le \alpha \le 1$, be covered by the mesh $\mathcal{T}_2^h$. Consider the set $Q^h_{p,-1}(\mathcal{T}_2^h)$. Let $G(x_1,x_2) = G(x_1) = x_1^{p+1} + f(x_1)$ where f is a polynomial of degree $\le p$. Then for $i = 1,2$

$$(5.7)\qquad Z_i = \inf_{\chi \in Q^h_{p+1-i,-1}} \left[\sum_{\Omega^h \in \mathcal{T}_2^h} \|G-\chi\|^2_{2,\Omega^h} \right]^{1/2} \ne o(h^{p-i}) .$$

Proof. Since $\mathcal{T}_2^h$ is a uniform mesh, the following inverse inequality holds on every element Ω^h for any polynomial $v \in Q^h_{p+1,-1}$

$$(5.8)\qquad \|v\|_{s,\Omega^h} \le Ch^{t-s}\|v\|_{t,\Omega^h}, \qquad 0 \le t \le s,$$

where C is independent of v, Ω^h, t, s, h. Suppose for $i = 1,2$

$$Z_i = o(h^{p-i})$$

and that this infimum is attained by $\chi_i \in Q^h_{p+1-i,-1}$. Then we have, using (5.8) with $v = G - \chi_i \in Q^h_{p+1,-1}$ and $s = p+2-i$, $t = 2$

$$\left[\sum_{\Omega^h \in \mathcal{T}_2^h} \|G-\chi_i\|^2_{p+2-i,\Omega^h}\right]^{1/2} \le Ch^{2-(p+2-i)} Z_i = o(1)$$

from which

$$\left[\sum_{\Omega^h \in \mathcal{T}_2^h} \left\| x_1^{i-1} + f^{(p+2-i)}(x_1) - \frac{\partial^{p+2-i}\chi_i}{\partial x_1^{p+2-i}} \right\|_{0,\Omega^h}\right]^{1/2} = o(1)$$

But this is impossible, since $\chi_i \in Q^h_{p+1-i,-1}$ implies $\frac{\partial^{p+2-i}\chi_i}{\partial x_i{}^{p+2-i}} = 0$ over each Ω^h. This proves (5.7). $\square$

We now prove the following lemma, using an idea from the proof of Theorem 2 in [7].

<u>Lemma 5.2</u>. Let $\Omega = \Omega_1$ be as in Lemma 5.1 and let $D(x_1,x_2) = x_1^{p+3}x_2^2$. Let

$$W_1 = (P^h_{p+1,1} \oplus Q^h_{p,1})(\mathcal{T}^h_2)$$

$$W_2 = P^h_{p+1,1}(\mathcal{T}^h_2)\ .$$

Then for $i = 1,2$

$$\inf_{\chi\in W_i} \|D-\chi\|_{2,\Omega} \neq o(h^{p-i})\ .$$

<u>Proof</u>. We use the standard notation

$$(x-\bar{x})^s_+ = (x-\bar{x})^s \qquad \text{for} \quad x \ge \bar{x}$$

$$= 0 \qquad \text{for} \quad x < \bar{x}\ .$$

Let $M = h^{-1}$ and for $0 \le i,j \le M-1$, let $(x^i_1, x^j_2) \equiv (ih, jh)$. Then any $\chi \in P^h_{p+1,-1}$ can be represented as a linear combination of the truncated powers

(5.9) $$(x_1-x^i_1)^{p_i}_+\ (x_2-x^j_2)^{q_j}_+ \qquad 0 \le i,j \le M-1$$

where $p_i,q_j \ge 0$ and $p_i + q_j \le p + 1$. Any $\chi \in Q^h_{p,-1}$ can be similarly represented by taking $p_i \le p$, $q_j \le p$. If we impose the constraints $p_i \ge 2$ for $i > 0$ and $q_j \ge 2$ for $j > 0$, then (5.9) gives us precisely all $\chi \in P^h_{p+1,1}$ and $Q^h_{p,1}$ respectively. Consequently, in this case, the only terms in $P^h_{p+1,1}$ of the form (5.9) with $p_i \ge p$ can be $(x_1-x^i_1)^{p+1}_+$, $(x-x^i_1)^p_+$ or $(x-x^i_1)^p x_2$ while those satisfying $q_j \ge p$ must be of the form $(x-x^j_2)^{p+1}_+$, $(x-x^j_2)^p_+$ or $x_1(x-x^j_2)^p_+$.

For $H > 0$, define the difference operator Δ_H by

$$\Delta_H g(x_1,x_2) = \sum_{l,m=0}^{2} (-1)^{l+m}\begin{pmatrix}2\\l\end{pmatrix}\begin{pmatrix}2\\m\end{pmatrix} g(x_1+la,\ x_2+ma)$$

where $a = H/4$. Let $H = sh$ where s is a positive integer. Then for any $\chi \in P^h_{p+1,1}$ (of the form (5.9)) with $p_i \geq p$ or $q_j \geq p$, we have

$$\Delta_H \chi = 0 .$$

This implies that $\Delta_H W_1 \subset Q^h_{p,1}(\mathcal{T}^h_2)$ and $\Delta_H W_2 \subset Q^h_{p-1,1}(\mathcal{T}^h_2)$. Also, we see that

$$\Delta_H D = Cx_1^{p+1} + f(x_1)$$

where $f(x_1)$ is a polynomial of degree $\leq p$. Hence, if we fix H to be sufficiently small and let $s \to \infty$ (i.e., $h \to 0$), we see by Lemma 5.1 that for $i = 1,2$

$$\inf_{\chi \in W_i} \|\Delta_H(D-\chi)\|_{2,\Omega_{1/2}} \geq C \inf_{\chi \in Q_{p+1-i,1}} \|x_1^{p+1} + f(x_1) - \chi\|_{2,\Omega_{1/2}} \neq o(h^{p-i}) .$$

Noting that

$$\|\Delta_H(D-\chi)\|_{2,\Omega_{1/2}} \leq C\,\|D-\chi\|_{2,\Omega}$$

(since Δ_H just gives a linear combination) proves the lemma. □

We can now prove theorems for rectangular meshes analogous to some theorems proven for triangular meshes.

<u>Theorem 5.5</u>. Let the extension procedure $\mathcal{F}$ consist of the h-version on the uniform mesh $\mathcal{T}^h_2$ for elements of type Q_p with $p = 1$. Then with respect to the solution set $H_{k,\nu}$, $k \geq 2$, $\mathcal{F}$ shows locking of order $N^{1/2}$, i.e., it is not robust.

Proof. The proof is essentially identical to that of Theorem 5.1. We now use Theorem 1 (instead of Theorem 3) of [7] to characterize the density of the space $P^h_{2,1}(\mathcal{T}^h_2) \supset \left[P^h_{2,1} \cup Q^h_{1,1}\right](\mathcal{T}^h_2)$. By this theorem, $P^h_{s,1}(\mathcal{T}^h_2)$ is dense in $C(\mathbb{R}^2)$ only when $s \geq 4$. □

Theorem 5.6. Let the extension procedure $\mathcal{F}$ consist of the h-version on the uniform mesh $\mathcal{T}^h_2$ for elements of type $\vec{Q}_p$ with $p \geq 2$. Then with respect to the solution set $H_{k,\nu}$, the extension $\mathcal{F}$ is uniformly robust with order $N^{-(p-1)/2}$, for $k \geq p$ and shows locking of $O(N^{1/2})$ for $k \geq p+1$.

Proof. By the results in [14], the optimal rate of convergence is obtained when $C^{(1)}$ tensor product splines of degree ≥ 2 in each variable are used. More precisely, we have for $\Phi \in H^{k+1}(\Omega)$, $k \geq p$,

$$\inf_{\chi \in W(\vec{Q}^h_{p,0}) = P^h_{p+1,1} \cup Q^h_{p,1}} \|\Phi - \chi\|_{2,\Omega} \leq \inf_{\chi \in Q^h_{p,1}} \|\Phi - \chi\|_{2,\Omega}$$

$$\leq Ch^{p-1} \|\Phi\|_{k+1,\Omega} .$$

Moreover, by Lemma 5.2, this is the best rate possible. The theorem follows, using Theorems 3.4 and 4.1. □

We now consider an extension procedure based on $\mathcal{T}^2_h$ using elements of type $\vec{Q}_p'$.

Theorem 5.7. Let the extension procedure $\mathcal{F}$ consist of the h-version on the uniform mesh $\mathcal{T}^h_2$ for elements of type Q_p'. Then with respect to the solution set $H_{k,\nu}$,

i) For $p = 1$, $k \geq 2$, $\mathcal{F}$ shows locking of $O(N^{1/2})$ and hence is non-robust.

ii) For $p = 2$, $\mathcal{F}$ is uniformly robust with order $N^{-1/2}$ for $k \geq 2$

and shows locking of order $N^{1/2}$ for $k \geq 3$.

iii) For $p > 2$, $\mathcal{T}$ is uniformly robust with order $N^{-(p-2)/2}$ for $k > p+2$ and shows locking of order N for $k \geq p+1$.

Proof. For $p \leq 2$, as shown in Section 4,

$$W(\vec{Q}^h_{p,0}{}') = W(\vec{Q}^h_{p,0}).$$

Hence, by Theorem 3.4, the results for locking and robustness are identical for $\vec{Q}_p{}'$ and $\vec{Q}_p$ elements, so that (i) and (ii) follow from Theorems 5.5 and 5.6, respectively.

By (4.9) and Theorem 3.4, we see that for $p > 2$, $\mathcal{T}$ is robust with uniform order

$$\tilde{g}(N) = \sup_{\Phi \in H^{(\alpha B)}_{k+1}(\Omega)} \inf_{\chi \in P^h_{p+1,1}(\mathcal{T}^h_2)} \|\Phi - \chi\|_{2,\Omega}.$$

By Theorem 2 of [7], we see that

$$\inf_{\chi \in P^h_{p+1,1}(\mathcal{T}^h_2)} \|\Phi - \chi\|_{C(\Omega)} \leq Ch^p \|\Phi\|_{C^{(p+2)}(\Omega)}$$

from which, using the inverse inequality and the fact that $\|\Phi\|_{C^{(p+2)}(\Omega)} \leq C\|\Phi\|_{k+1,\Omega}$ we obtain

(5.10) $$\tilde{g}(N) \leq CN^{-(p-2)/2}.$$

Moreover, by Lemma 5.2, this is the best estimate possible. This establishes (iii). □

Comparing elements of types Q_p and $Q_p{}'$, we see that both show locking but that the locking shown by $Q_p{}'$ elements is twice a strong as that of Q_p.

Let us briefly discuss meshes $\mathcal{T}^h_4$ and $\mathcal{T}^h_5$ using Q_p elements. For the mesh $\mathcal{T}^h_4$, the statement about robustness $\left[O\left(N^{-(p-1)/2}\right)\right]$ is exactly the

same as for the mesh $\mathcal{T}_2^h$. Nevertheless, an exact assessment of the locking is not presented here.

In the case of the meshes $\mathcal{T}_5^h$, it is possible to generalize the approach of the construction of Bogner-Fox-Schmidt rectangles (see [13]) to once more prove robustness of $O\left(N^{-(p-1)/2}\right)$, provided $p \ge 4$. We do not discuss the case $p < 4$ here.

Let us summarize the results of this section:

Type of mesh	Type of element	Order of locking, r $f(N)=O(N^r)$	Robustness order, q $g(N)=O(N^{-q})$
Uniform triangular	P_p, $p = 1$	$r = 1/2$	$q = 0$
	$p = 2$	$r = 1/2$	$q = 1/2$
	$p = 3$	$r = 1/2$	$q = 1$
Quasiuniform triangular	P_p, $p \ge 4$	$r = 0$	$q = p/2$
Uniform rectangular	Q_1, Q_1'	$r = 1/2$	$q = 0$
	Q_p, $p \ge 2$	$r = 1/2$	$q = (p-1)/2$
	Q_p', $p = 2$	$r = 1/2$	$q = 1/2$
	Q_p', $p \ge 3$	$r = 1$	$q = (p-2)/2$
Quasiuniform rectangular	Q_p, $p \ge 2$	$0 \le r \le \frac{1}{2}$	$q = (p-1)/2$
Quasiuniform parallelogram	Q_p, $p \ge 4$	$0 \le r \le \frac{1}{2}$	$q = (p-1)/2$

6. The p and h-p versions.

In [23], it was shown that the p-version (using straight-sided

triangular meshes) for (3.1) leads to a robust estimate of $O\left(N^{-\frac{(k-1)}{2}+\varepsilon}\right)$, $\varepsilon > 0$ arbitrary, when the solution is known to lie in $H^k(\Omega)$, i.e., the order of locking is at most N^{ε}. The loss in order ε resulted due to the definition of locking employed (see Remark 3.1) and is directly related to the locking in the pressures. Using the theorems from Section 3, we are able to obtain an optimal robustness estimate and prove that there is no locking for the displacements in the sense of Definition 3.1. This holds for parallelogram elements as well. Using this approach, we remove the dependence on ε that will occur in the robustness estimate for the displacements if the results from [23] are used.

Theorem 6.1. Let the extension procedure $\mathcal{F}$ consist of the p-version using a mesh consisting of triangles or parallelograms. Then with solution sets $H_{k,\nu}$, $k \geq 2$, $\mathcal{F}$ is free of locking and is robust with uniform order $N^{-(k-1)/2}$.

Proof. Using the results of [21] and [16], we have for the p-version with $C^{(1)}$ continuous triangular or parallelogram straight-sided elements (h_0 fixed), for $\phi \in H^{k+1}(\Omega)$, $k \geq 2$,

$$\inf_{\chi \in P^{h_0}_{p,1}} \|\phi - \chi\|_{2,\Omega} \leq Cp^{-(k-1)} \|\phi\|_{k+1,\Omega} \tag{6.1}$$

Since $P^{h_0}_{p,1} \subset W(\vec{V}^{h_0}_p)$, using Theorem 3.4 gives the required robustness rate. The absence of locking follows by Theorem 4.2. □

Using the results of [19], it can be shown that the h-p version with quasiuniform triangular meshes results in locking of order at most p^{ε}, with the rate in h being optimal, provided $p \geq 4$. We are able to prove the following theorem.

Theorem 6.2. Let the extension procedure $\mathcal{F}$ consist of the h-p version, using quasiuniform meshes consisting of triangles. Let the solution sets be $H_{k,\nu}$ with $k \geq 2$. Then for $p \geq \max(4, k-1)$, $\mathcal{F}$ is free of locking and is robust with uniform order $N^{-(k-1)/2}$ (or, more precisely $h^{k-1}p^{-(k-1)}$).

Proof. We only outline the proof here. Essentially, the idea is to use (6.1) together with a standard scaling argument to show that

$$\inf_{\chi \in P^h_{p,1}} \|\phi - \chi\|_{2,\Omega} \leq Ch^{k-1}p^{-(k-1)}\|\phi\|_{k+1,\Omega} \tag{6.2}$$

provided $p \geq k-1$. For details see [2], where this scaling argument has been used to prove an estimate analogous to (6.2) for the case of $C^{(0)}$ elements. Using Theorems 3.4, 4.2 (or Eqn. (4.3)) completes the proof. □

7. Some generalizations.

So far, we have only considered the two-dimensional isotropic case. It turns out that situations analogous to Poisson locking arise in more general contexts as well. For example, the same phenomenon is observed in the three-dimensional isotropic case when $\nu \rightarrow 0.5$. Moreover, one can consider similar situations that occur in the anisotropic case (both 2-D and 3-D), when locking may be observed due to the introduction of a constraint on the approximate subspaces.

Let us look at the general 3-D equations of anisotropic elasticity given by

$$A\sigma = \varepsilon(\vec{u}) \quad \text{in } \Omega \tag{7.1}$$

$$\operatorname{div} \sigma = \vec{f} \quad \text{in } \Omega \tag{7.2}$$

$$\sigma\vec{n} = \vec{g} \quad \text{on } \Gamma \tag{7.3}$$

where σ is the 3×3 symmetric stress tensor, and $\varepsilon(\vec{u})$ the strain tensor.

A is a fourth-order tensor known as the *compliance tensor* which depends upon the properties of the material. It is a self-adjoint linear operator acting on the six-dimensional space of symmetric 3x3 tensors, and is characterized by specifying 21 independent elastic moduli. $\vec{n}$ is the unit normal to the boundary as before and we assume (2.5) again.

The standard variational form of (7.1)-(7.3) is obtained under the assumption that A is positive definite, in which case (7.1) may be solved for σ and substituted in (7.2) to give a problem involving the unknown $\vec{u}$ alone. For many important materials, however, A may be positive semi-definite and singular (or close to singular). In this case, if $0 \le \lambda_1 \le \lambda_2 \dots \le \lambda_6$ denote its eigenvalues and $\sigma_1, \sigma_2, \dots \sigma_6$ a corresponding set of orthonormal eigenvectors, then $\lambda_1 = 0$ (or is close to 0). The corresponding constraint on $\vec{u}$, analogous to (1.1), becomes

$$\operatorname{div}(\sigma_1 \vec{u}) = 0 \tag{7.4}$$

In fact, Poisson locking is a special case of (7.4), because for the case of isotropic materials with $\nu = 0.5$, it can be shown that the identity matrix is an eigenvector of A corresponding to the eigenvalue 0.

There are two cases which must be distinguished between. (7.4) defines a *singular* constraint if σ_1 is a singular tensor. If σ_1 is non-singular, then (7.4) is called a *non-singular* constraint. Poisson locking is obviously an example of a *non-singular* constraint. On the other hand, a material that is inextensible in the direction $\vec{r}$ satisfies (7.4) with a singular σ_1 given by

$$\sigma_1 = \vec{r}^T \vec{r} \tag{7.5}$$

Both singular and non-singular constraints will lead to locking, when the material is such that A is (or is close to being) singular. Singular

constraints like (7.5) are analogous to the locking constraints found in shell problems and are generally harder to develop robust methods for. See Section 3 of [4], where we have discussed the robustness of various methods for the problems of heat transfer through highly anisotropic materials, which has the same character as locking due to a singular constraint.

Here, we are interested in the case when A is not close to having a singular constraint (this is characterized precisely in [1]) but *is* close to having a non-singular constraint like (7.4). Such a situation has been characterized in detail for the case of 3-D orthotropic materials in [1]. Denote by $\vec{u}_{\lambda_1}$ the solution to (7.1)-(7.3) when the compliance tensor A_{λ_1} has minimum eigenvalue $\lambda_1 > 0$ and let the limiting constrained case be A_0. Then by Theorem 1.1 of [1], if the constraint is non-singular, there exists a unique limit solution u_0 satisfying (7.4) with

$$\lim_{A_{\lambda_1} \to A_0} \|\vec{u}_{\lambda_1} - \vec{u}_0\|_{1,\Omega} = 0$$

It is expected that an analog of (2.12) (with λ_1 replacing $1-2\nu$) will be valid in this case as well. However, such a result is not currently available.

Suppose we discretize the standard variational formulation of (7.1)-(7.3) (obtained by first solving (7.1) for σ). Then, without establishing (2.12), Theorems 3.2 - 3.4 cannot be proved as stated. However, it is still instructive to look at the limit problem, where the exact and approximate solutions must satisfy the constraint (7.4). Suppose we are interested in exact solutions $\vec{u} \in H$, where H characterizes the smoothness. Let H_0 be the subset of H satisfying (7.4) and $Z(\vec{V}^N)$ the subset of $\vec{V}^N$ satisfying (7.4). Then the limit problem has the optimal rate of convergence $F_0(N)$ in the $\vec{H}^1(\Omega)$ norm if and only if

(7.6) $$g(N) = \sup_{\vec{u}\in H_0^B} \inf_{\vec{w}\in Z(\vec{V}^N)} \|\vec{u} - \vec{w}\|_{1,\Omega} \le CF_0(N).$$

By Theorems 2.2(A), 2.3(A) of [4], (7.6) is a *necessary* condition for $\mathcal{F}$ to be free of locking with respect to the $\vec{H}^1(\Omega)$ or the energy norm.

When σ_1 is non-singular, we can derive a more convenient condition which is *sufficient* for (7.6) to hold and is analogous to (3.11). As in the two-dimensional case, we have

$$\operatorname{div}\vec{z} = 0 \Rightarrow \vec{z} = \operatorname{Curl}\vec{\phi}.$$

Hence, for $\vec{u} \in H_0^B$, $\vec{w} \in Z(V^N)$, we have

$$\sigma_1\vec{u} = \operatorname{Curl}\vec{\phi}, \qquad \sigma_1\vec{w} = \operatorname{Curl}\vec{\chi}$$

where $\vec{\phi} \in W(H_0^B)$, $\vec{w} \in W(V^N)$. Here,

(7.7) $$W(Y) = \{\vec{\psi},\ \sigma_1^{-1}\operatorname{Curl}\vec{\psi} \in Y\}.$$

Then we have

$$\|\vec{u} - \vec{w}\|_1 = \|\sigma^{-1}\operatorname{Curl}(\vec{\phi} - \vec{\chi})\|_{1,\Omega} \le \|\vec{\phi} - \vec{\chi}\|_{2,\Omega}$$

so that the following is sufficient for (7.6) to hold:

(7.8) $$\sup_{\vec{\phi}\in W(H_0^B)} \inf_{\vec{\chi}\in W(V^N)} \|\vec{\phi} - \vec{\chi}\|_{2,\Omega} \le CF_0(N).$$

As an application of (7.8), let us look at the case that $\{\vec{V}^N\}$ consists of continuous piecewise polynomials of fixed degree p on a quasiuniform sequence of meshes consisting of tetrahedra. Then for any non-singular σ_1, σ_1^{-1} will be bijective on $\vec{V}^N$, so that from (7.7), we see that

$$W(\vec{V}^N) = W(\vec{V}_p^h) = P_{p+1,1}^h$$

analogously to (4.7). By [24], the minimum q required to obtain an optimal

rate of convergence for the h-version using $P^h_{q,1}$ is $q = 9$, so that (7.6) will be satisfied whenever $\vec{V}^N = \vec{V}^h_p$ with $p \geq 8$ and the h-version for the limit problem will show optimal convergence. We expect once again (similar to the results in Section 5) that the use of lower p or the use of parallelepiped elements will result in locking.

Acknowledgement. The authors would like to thank Prof. C. de Boor for his valuable communications regarding approximation orders, particularly references [6] and [10].

References.

[1] D.N. Arnold and R.S. Falk, Well-posedness of the fundamental boundary value problems for constrained anisotropic elastic materials, Archive for Rational Mechanics and Analysis **98** (1987), 143-165.

[2] I. Babuška and M. Suri, The h-p version of the finite element method with quasiuniform meshes, RAIRO Math. Mod. Num. Anal. **21** (1987), 199-238.

[3] I. Babuška and M. Suri, The optimal convergence rate of the p-version of the finite element method, SIAM J. Numer. Anal. **24** (1987), 750-776.

[4] I. Babuška and M. Suri, On locking and robustness in the finite element method, Tech. Rep. BN-1112, Institute for Physical Science and Technology, University of Maryland, College Park, May 1990.

[5] I. Babuška and M. Suri, The problem of Poisson ratio locking in curved elements, (to appear).

[6] C. de Boor, private communication, 1990.

[7] C. de Boor and R. DeVore, Approximation by smooth multivariate splines, Trans. Amer. Math. Soc. **276** (1983), 775-788.

[8] C. de Boor and K. Höllig, Approximation order from bivariate C^1 cubics: a counterexample, Proc. Amer. Math. Soc. **87** (1983), 649-655.

[9] C. de Boor and K. Höllig, Approximation power of smooth bivariate pp functions, Math. Z. **197** (1988), 343-363.

[10] C. de Boor and R.Q. Jia, A sharp upper bound on the approximation order of smooth bivariate pp functions, to appear in the *Journal for Approximation Theory.*

[11] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, to appear.

[12] C.K. Chui, *Multivariate Splines*, SIAM, Philadelphia (1988).

[13] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[14] G. Fix and G. Strang, Fourier analysis of the fintie element method in Ritz-Galerkin theory, Studies in Appl. Math. **48** (1969), 265-273.

[15] T.J. Hughes, *The Finite Element Method*, Prentice Hall, 1987.

[16] S. Jensen, An H_0^m Interpolation Result, SIAM J. Math. Anal. 22, May, 1991.

[17] N.I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, P. Noordhoff Ltd., Groningen, The Netherlands, 1963.

[18] A. Pinkus, *n-widths in Approximation Theory*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo (1985).

[19] L.R. Scott and M. Vogelius, Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials, RAIRO Math. Modeling Num. Anal. **19** (1985), 111-143.

[20] L.R. Scott and M Vogelius, Conforming finite element methods for incompressible and nearly incompressible continua, *Large Scale Computations in Fluid Mechanics*, Lectures in Applied Mathematics, Volume 22, Part 2, 221-244, AMS, Providence, R.I. (1985).

[21] M. Suri, The p-version of the finite element method for elliptic equations of order 2ℓ, RAIRO Math. Modeling Num. Anal. (1990), 265-304.

[22] B.A. Szabo and I. Babuška, *Finite Element Analysis*, John Wiley and Sons, Inc., 1991 (in press).

[23] M. Vogelius, An analysis of the p-version of the finite element method for nearly incompressible materials. Uniformly valid, optimal error estimates, Numer. Math. **41** (1983), 19-37.

[24] A. Zenisek, Polynomial approximation on tetrahedrons in the finite element method, J. Approximation Theory 7 (1973), 334-351.

The Laboratory for Numerical Analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

- To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.

- To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.

- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

- To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Institute of Standards and Technology.

- To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.).

Further information may be obtained from **Professor I. Babuska**, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.